

A class of cyclic Codes Over the Ring $\mathbb{Z}_4[u]/\langle u^2 \rangle$ and its Gray image

Sukhamoy Pattanayak and Abhay Kumar Singh

Department of Applied Mathematics

Indian School of Mines

Dhanbad 826 004, India

e-mail : sukhamoy88@gmail.com

singh.ak.am@ismdhanbad.ac.in

Abstract

Cyclic codes over R have been introduced recently. In this paper, we study the cyclic codes over R and their \mathbb{Z}_2 image. Making use of algebraic structure, we find the some good \mathbb{Z}_2 codes of length 28.

1 Introduction

The study of cyclic codes over finite ring is a topic of growing interest due to their rich algebraic structure. In the seminal paper [3], Hammons et al. established connection between some good binary nonlinear codes and \mathbb{Z}_4 -linear codes via gray map. The different aspect of cyclic codes over \mathbb{Z}_4 and some other finite rings have been discussed in series of papers[1,11-13]. In[15], Yildiz and Aydin discussed the cyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ and their \mathbb{Z}_4 image and added some good codes to the data base of \mathbb{Z}_4 codes. The cyclic codes of odd length over $\mathbb{Z}_4 + u\mathbb{Z}_4$ are studied in[10]. Yildiz and karadeniz discussed the linear codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ in[2].

The ring $\mathbb{Z}_4 + u\mathbb{Z}_4$ is extension of \mathbb{Z}_4 and it is a frobenius ring. We extended the result of Yildiz and Aydin by studying the \mathbb{Z}_2 image of $\mathbb{Z}_4 + u\mathbb{Z}_4$. The class of quasi-cyclic codes(QC) is a great platform of obtaining good codes which attain a version of Gilbert Varshamov bound. These finding motivated the study of cyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ and their \mathbb{Z}_2 gray image, because as we can directly be shown, the \mathbb{Z}_2 images are equivalent to 4-QC codes over \mathbb{Z}_2 . With the help of example, we find some good \mathbb{Z}_2 codes through cyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$.

This paper is organized as follows. In section 2, we have given some preliminaries related with this work. In sec 3, we find gray image of cyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$. We have reviewed some results of cyclic codes of $\mathbb{Z}_4 + u\mathbb{Z}_4$ in sec 4. In sec 5, we have provided some good \mathbb{Z}_2 codes of length 28.

2 Preliminaries

Let R be the commutative, characteristic 4 ring $\mathbb{Z}_4 + u\mathbb{Z}_4 = \{a + ub | a, b \in \mathbb{Z}_4\}$ with $u^2 = 0$. R can also be thought of as the quotient ring $\mathbb{Z}_4[u]/\langle u^2 \rangle$. The units of R are

$$1, 3, 1 + u, 1 + 2u, 1 + 3u, 3 + u, 3 + 2u, 3 + 3u,$$

and the non-units are

$$0, 2, u, 2u, 2 + u, 2 + 2u, 3u, 2 + 3u.$$

R has six ideals in all listed below:

$$\{0\}, \langle u \rangle, \langle 2 \rangle, \langle 2u \rangle, \langle 2 + u \rangle, \langle 2, u \rangle.$$

R is a non-principal local ring with $\langle 2, u \rangle$ as its unique maximal ideal. A commutative ring is called a chain ring if its ideals form a chain under the relation of inclusion. But from the above we see that the ideals of R do not form chain. Therefore, R is a non-chain ring. A linear code C of length n over R is a R -submodule of R^n . An element of C is called a codeword. A code of length n is cyclic if the code is invariant under the automorphism σ which has

$$\sigma(c_0, c_1, \dots, c_{n-1}) = (c_{n-1}, c_0, \dots, c_{n-2}).$$

A code of length n is 4-quasi cyclic if the code is invariant under the automorphism ν which has

$$\nu(c_0, c_1, \dots, c_{n-4}, c_{n-3}, c_{n-2}, c_{n-1}) = (c_{n-4}, c_{n-3}, c_{n-2}, c_{n-1}, c_0, \dots, c_{n-5}).$$

It is well known that a cyclic code of length n over R can be identified with an ideal in the quotient ring $R[x]/\langle x^n - 1 \rangle$ via the R -module isomorphism as follows:

$$\begin{aligned} R^n &\longrightarrow R[x]/\langle x^n - 1 \rangle \\ (c_0, c_1, \dots, c_{n-1}) &\mapsto c_0 + c_1x + \dots + c_{n-1}x^{n-1} \pmod{\langle x^n - 1 \rangle} \end{aligned}$$

The residue field is given by $R/\langle 2, u \rangle \cong F_2$. The image of any element $a \in R$ under the projection map $\mu : R \longrightarrow \overline{R}$ is denoted by \overline{a} . The map μ is extended to $R[x] \longrightarrow \overline{R}[x]$ in the natural way. A polynomial $f(x) \in R[x]$ is called basic irreducible (primitive) if $f(x)$ is an irreducible (primitive) polynomial in $R[x]$. A polynomial $f(x)$ over R is called a regular polynomial if it is not a zero divisor in $R[x]$, equivalently, $f(x)$ is regular if $f(x) \neq 0$. Two polynomials $f(x), g(x) \in R[x]$ are said to be coprime if there exist $a(x), b(x) \in R[x]$ such that $a(x)f(x) + b(x)g(x) = 1$.

Theorem 2.1 (*Hensel's Lemma[13]*) *Let f be a monic polynomial in $\mathbb{Z}_4[x]$ and assume that $f(\text{mod } 2) = g_1g_2 \cdots g_r$, where g_1, g_2, \dots, g_r are pairwise coprime monic polynomials over F_2 . Then there exist pairwise coprime monic polynomials f_1, f_2, \dots, f_r over \mathbb{Z}_4 such that $f = f_1f_2 \cdots f_r$ in $\mathbb{Z}_4[x]$ and $f_i(\text{mod } 2) = g_i, i = 1, 2, \dots, r$.*

3 Gray images of cyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$

The Lee weight was defined as $w_L(a) = \min\{a, 4 - a\}$, $a \in \mathbb{Z}_4$ i.e, $w_L(0) = 0, w_L(1) = 1, w_L(2) = 2, w_L(3) = 1$. Let $a + ub$ be any element of the ring $R = \mathbb{Z}_p + u\mathbb{Z}_p, u^2 = 0$. The Lee Weight w_L of the ring R is defined as follows

$$w_L(a + ub) = w_L((b, a + b)),$$

where $w_L((b, a + b))$ described the usual Lee weight on \mathbb{Z}_4^2 . For any $c_1, c_2 \in R$, the Lee distance d_L , given by $d_L(c_1, c_2) = w_L(c_1 - c_2)$. The minimum Lee distance of C is the smallest nonzero Lee distance between all pairs of distinct codewords of C . The Hamming weight of a codeword $c = (c_0, c_1, \dots, c_{n-1})$ denoted by $w_H(c)$ is the number of non zero entries in c . The Hamming distance $d(c_1, c_2)$ between two codewords c_1 and c_2 is the Hamming weight of the codeword $c_1 - c_2$. The minimum Hamming distance of a linear code C is given by

$$d_H(C) = \min\{d(c_1, c_2) : c_1, c_2 \in C, c_1 \neq c_2\}.$$

Now we define the Gray map on R . Any element $c \in R$ can be expressed as $c = a + ub$, where $a, b \in \mathbb{Z}_4$. The Gray map defined as follows

$$\begin{aligned} \phi : \mathbb{Z}_4 + u\mathbb{Z}_4 &\longrightarrow \mathbb{Z}_4^2 \\ \text{such that } \phi(a + ub) &= (b, a + b) \quad a, b \in \mathbb{Z}_4 \end{aligned}$$

Again we give the definition of the Gray map from \mathbb{Z}_4 to \mathbb{Z}_2^2 . First we see that the 2-adic expansion of $c \in \mathbb{Z}_4$ is $c = \alpha(c) + 2\beta(c)$ such that $\alpha(c) + \beta(c) + \gamma(c) = 0$ for all $c \in \mathbb{Z}_4$.

Then we get the table below

c	$\alpha(c)$	$\beta(c)$	$\gamma(c)$
0	0	0	0
1	1	0	1
2	0	1	1
3	1	1	0

The Gray map $\psi : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2^2$ given by $\psi(c) = (\beta(c), \gamma(c))$, $c \in \mathbb{Z}_4$ i.e, $\psi(0) = (0, 0), \psi(1) = (0, 1), \psi(2) = (1, 1), \psi(3) = (1, 0)$. Then we get the composite map as follows

$$\begin{aligned} \Phi(\psi \cdot \phi) : R &\longrightarrow \mathbb{Z}_2^4 \\ \Phi(a + ub) &= \psi(\phi(a + ub)) = \psi(b, a + b) \\ &= (\beta(b), \gamma(b), \beta(a + b), \gamma(a + b)). \end{aligned}$$

It is well known that Φ is a linear map on R and is an isometry from $(R, \text{Lee distance})$ to $(\mathbb{Z}_2^4, \text{Hamming distance})$. This map Φ can be extended to R^n in the natural way:

$$\begin{aligned} \Phi : R^n &\longrightarrow \mathbb{Z}_2^{4n} \\ (c_0, c_1, \dots, c_{n-1}) &\longrightarrow (\beta(b_0), \gamma(b_0), \beta(a_0 + b_0), \gamma(a_0 + b_0), \beta(b_1), \gamma(b_1), \beta(a_1 + b_1), \\ &\quad \gamma(a_1 + b_1), \dots, \beta(b_{n-1}), \gamma(b_{n-1}), \beta(a_{n-1} + b_{n-1}), \gamma(a_{n-1} + b_{n-1})) \end{aligned}$$

where $c_i = a_i + ub_i, 0 \leq i \leq n - 1$.

Proposition 3.1 *The Gray map Φ is a distance-preserving map from $(R^n, \text{Lee distance})$ to $(\mathbb{Z}_2^{4n}, \text{Hamming distance})$ and this map also \mathbb{Z}_2 linear.*

Proof : From the definitions, it is clear that $\Phi(c_1 - c_2) = \Phi(c_1) - \Phi(c_2)$ for $c_1, c_2 \in R^n$. Thus, $d_L(c_1, c_2) = w_L(c_1 - c_2) = w_H(\Phi(c_1 - c_2)) = w_H(\Phi(c_1) - \Phi(c_2)) = d_H(\Phi(c_1), \Phi(c_2))$. Let $c_1, c_2 \in R^n, k_1, k_2 \in \mathbb{Z}_2$, then from the definition of the Gray map, we have $\Phi(k_1 c_1 + k_2 c_2) = k_1 \Phi(c_1) + k_2 \Phi(c_2)$, that implies Φ is \mathbb{Z}_2 linear.

Theorem 3.1 *Let σ be the cyclic shift of R^n and ν denote the 4-QC shift of \mathbb{Z}_2^{4n} . Let Φ be the Gray map from R^n to \mathbb{Z}_2^{4n} . Then prove that $\Phi\sigma = \nu\Phi$*

Proof : Let $c = (c_0, c_1, \dots, c_{n-1}) \in R^n$, where $c_i = a_i + ub_i$, with $a_i, b_i \in \mathbb{Z}_4, 0 \leq i \leq n-1$. From the definition of the Gray map, we get

$$\Phi(c) = (\beta(b_0), \gamma(b_0), \beta(a_0 + b_0), \gamma(a_0 + b_0), \beta(b_1), \gamma(b_1), \beta(a_1 + b_1), \gamma(a_1 + b_1), \dots, \beta(b_{n-1}), \gamma(b_{n-1}), \beta(a_{n-1} + b_{n-1}), \gamma(a_{n-1} + b_{n-1}))$$

Hence

$$\nu(\Phi(c)) = (\beta(b_{n-1}), \gamma(b_{n-1}), \beta(a_{n-1} + b_{n-1}), \gamma(a_{n-1} + b_{n-1}), \beta(b_0), \gamma(b_0), \beta(a_0 + b_0), \gamma(a_0 + b_0), \dots, \beta(b_{n-2}), \gamma(b_{n-2}), \beta(a_{n-2} + b_{n-2}), \gamma(a_{n-2} + b_{n-2})).$$

On the other hand,

$$\sigma(c) = (c_{n-1}, c_0, c_1, \dots, c_{n-2})$$

we deduce that

$$\Phi(\sigma(c)) = (\beta(b_{n-1}), \gamma(b_{n-1}), \beta(a_{n-1} + b_{n-1}), \gamma(a_{n-1} + b_{n-1}), \beta(b_0), \gamma(b_0), \beta(a_0 + b_0), \gamma(a_0 + b_0), \dots, \beta(b_{n-2}), \gamma(b_{n-2}), \beta(a_{n-2} + b_{n-2}), \gamma(a_{n-2} + b_{n-2})).$$

Therefore $\Phi\sigma = \nu\Phi$.

Theorem 3.2 *A linear code C of length n over R is a cyclic code if and only if $\Phi(C)$ is a 4-quasi cyclic code of length $4n$ over \mathbb{Z}_2 .*

Proof : It is immediately get from previous theorem.

Corollary 3.1 *The Gray image of a cyclic code of length n over R is a distance invariant linear 4-quasi cyclic code of length $4n$ over \mathbb{Z}_2 .*

Proof : Let C be a cyclic code of length n over R . Then $\sigma(C) = C$, therefore $\Phi(\sigma(C)) = \Phi(C)$. It follows from theorem 3.1 that $\nu(\Phi(C)) = \Phi(C)$, which means that $\Phi(C)$ is a 4-quasi cyclic code.

4 Cyclic codes

Let length n is odd through out that section. For a finite chain ring \mathbb{R} , it is well known that the ring $\frac{\mathbb{R}[x]}{\langle x^n - 1 \rangle}$ is a principal ideal ring. But in that case the ring $R = \mathbb{Z}_4 + u\mathbb{Z}_4, u^2 = 0$ is not a chain ring and the situation is not so easy.

Proposition 4.1 ([10]) *The ring $R_n = \frac{R[x]}{\langle x^n - 1 \rangle}$ is not a principal ideal ring.*

Therefore, a cyclic code of length n over R is not principally generated. As n is odd, the ring $\frac{\mathbb{Z}_4[x]}{\langle x^n-1 \rangle}$ is a principal ideal ring. So a cyclic code of length n over R is of the form $C = C_1 + uC_2 = \langle g_1 \rangle + u\langle g_2 \rangle$, where $g_1, g_2 \in \mathbb{Z}_4[x]$ are generator polynomials of the cyclic codes C_1, C_2 , respectively.

We assume the ideals of $R[x]/\langle g \rangle$, where g is a basic irreducible polynomial over R .

Theorem 4.1 *If $g \in R[x]$ be a basic irreducible polynomial. Then the ideals of $R[x]/\langle g \rangle$ are precisely, $\{0\}, \langle 1 + \langle g \rangle \rangle, \langle 2 + \langle g \rangle \rangle, \langle u + \langle g \rangle \rangle, \langle 2u + \langle g \rangle \rangle, \langle 2 + u + \langle g \rangle \rangle$ and $\langle \langle 2, u \rangle + \langle g \rangle \rangle$.*

Theorem 4.2 *Let $x^n - 1 = g_1 g_2 \cdots g_m$, where $g_i, i = 1, 2, \dots, m$ are basic irreducible pairwise coprime polynomials in $R[x]$. Then any ideal in R_n is the sum of the ideals of $R[x]/\langle g_i \rangle, i = 1, 2, \dots, m$.*

Proof : It follows from the Chinese Remainder Theorem.

Corollary 4.1 *There are 7^m cyclic codes of length n over R .*

Theorem 4.3 *A linear code $C = C_1 + uC_2$ of length n over R is cyclic if and only if C_1, C_2 are cyclic codes of length n over \mathbb{Z}_4 .*

Define $\psi : R \longrightarrow \mathbb{Z}_4$ by $\psi(a + ub) = a \pmod{u}$, where $a, b \in \mathbb{Z}_4$. The map ψ is a ring homomorphism. We extend the map ψ to a homomorphism $\phi : \frac{R[x]}{\langle x^n-1 \rangle} \longrightarrow \frac{\mathbb{Z}_4[x]}{\langle x^n-1 \rangle}$ defined by

$$\phi(c_0 + c_1x + \cdots + c_{n-1}x^{n-1}) = \psi(c_0) + \psi(c_1)x + \cdots + \psi(c_{n-1})x^{n-1},$$

where $c_i \in R$. We have $\ker \phi = \langle u \rangle = u\mathbb{Z}_4$.

Let C be a cyclic code of length n over R . Restrict ϕ to C and define

$$J = \{h(x) \in \frac{\mathbb{Z}_4[x]}{\langle x^n-1 \rangle} : uh(x) \in \ker \phi\}.$$

Obviously J is an ideal of $\frac{\mathbb{Z}_4[x]}{\langle x^n-1 \rangle}$. So J is a cyclic code over \mathbb{Z}_4 . Again, the image of C under ϕ is an ideal of $\frac{\mathbb{Z}_4[x]}{\langle x^n-1 \rangle}$.

Let n be an odd integer. Then the cyclic code of length n over \mathbb{Z}_4 is principally generated. So $\phi(C)$ and $\ker \phi$ are principal ideals of $\frac{\mathbb{Z}_4[x]}{\langle x^n-1 \rangle}$, and $\phi(C) = \langle f_1(x) + 2f_2(x) \rangle$ and $\ker \phi = \langle uf_3(x) + 2uf_4(x) \rangle$, where $f_2(x)|f_1(x)|x^n-1$ and $f_4(x)|f_3(x)|x^n-1$. Therefore $C = \langle f_1(x) + 2f_2(x) + uf_{13}(x) + 2uf_{14}(x), uf_3(x) + 2uf_4(x) \rangle$. Using the theorem which is used in [2], we get the following result.

Theorem 4.4 *Let n be an odd integer and C be a cyclic code of length n over R . Then*

$$C = \langle f_1(x) + 2f_2(x) + 2uf_{14}(x), uf_3(x) + 2uf_4(x) \rangle,$$

where $f_2(x)|f_1(x)|x^n-1$ and $f_4(x)|f_3(x)|x^n-1$ in $\frac{R[x]}{\langle x^n-1 \rangle}$.

We Know that the basis of C over R is called the minimal generating set of C , and the number of element include in the minimal generating set is called the rank of the code C , denoted the $rank(C)$.

Theorem 4.5 *Let C be a cyclic code of length n over R . If $C = \langle f_1(x) + 2f_2(x) + 2uf_{14}(x), uf_3(x) + 2uf_4(x) \rangle$ and $\deg f_1(x) = k_1$ and $\deg f_2(x) = k_2$, then C has rank $n - k_2$ and a minimal spanning set $A = \{(f_1(x) + 2f_2(x) + 2uf_{14}(x)), x(f_1(x) + 2f_2(x) + 2uf_{14}(x)), \dots, x^{n-k_1-1}(f_1(x) + 2f_2(x) + 2uf_{14}(x)), u(f_3(x) + 2f_4(x)), xu(f_3(x) + 2f_4(x)), \dots, x^{k_1-k_2-1}u(f_3(x) + 2f_4(x))\}$.*

5 Example

In this section, we give some examples of cyclic codes of different lengths over the ring R .

Example 5.1 *Cyclic codes of length 3 over $R = \mathbb{Z}_4 + u\mathbb{Z}_4, u^2 = 0$: We have*

$$x^3 - 1 = (x - 1)(x^2 + x + 1) \text{ over } R.$$

Let $g_1 = x - 1$ and $g_2 = x^2 + x + 1$. The cyclic codes of length 3 over R are given in Table 1.

Table 1. Non-zero cyclic codes of length 3 over $\mathbb{Z}_4 + u\mathbb{Z}_4, u^2 = 0$.

Non-zero generator polynomials	ranks
$\langle 2g_i, ug_1 + 2u \rangle, i = 1, 2.$	2
$\langle 2g_i, ug_2 + 2u \rangle, i = 1, 2.$	1
$\langle 2g_i, 3u \rangle, i = 1, 2.$	3
$\langle 2, ug_1 + 2u \rangle$	2
$\langle 2, ug_2 + 2u \rangle$	1
$\langle 2, 3u \rangle$	3
$\langle g_1 + 2, 3u \rangle$	3
$\langle g_2 + 2, 3u \rangle$	3

Example 5.2 *Cyclic codes of length 7 over $R = \mathbb{Z}_4 + u\mathbb{Z}_4, u^2 = 0$: We have*

$$x^7 - 1 = (x - 1)(x^3 + x + 1)(x^3 + x^2 + 1) \text{ over } F_2.$$

This factors are irreducible polynomials over F_2 . The Hensel lifts of $x^3 + x + 1$ to \mathbb{Z}_4 is $x^3 + 2x^2 + x - 1$ and Hensel lifts of $x^3 + x^2 + 1$ to \mathbb{Z}_4 is $x^3 - x^2 - 2x - 1$. Therefore we have

$$x^7 - 1 = (x - 1)(x^3 + 2x^2 + x - 1)(x^3 - x^2 - 2x - 1) \text{ over } R.$$

Let $g_1 = x - 1$, $g_2 = x^3 + 2x^2 + x - 1$ and $g_3 = x^3 - x^2 - 2x - 1$. The cyclic codes of length 7 over R are given in Table 2.

Table 2. Non-zero cyclic codes of length 7 over $\mathbb{Z}_4 + u\mathbb{Z}_4, u^2 = 0$.

Non-zero generator polynomials	ranks
$\langle 2g_i g_j, u g_1 g_2 + 2u g_1 \rangle, i \neq j = 1, 2, 3.$	3
$\langle 2g_i g_j, u g_1 g_2 + 2u g_2 \rangle, i \neq j = 1, 2, 3.$	3
$\langle 2g_i g_j, u g_1 g_2 + 2u \rangle, i \neq j = 1, 2, 3.$	3
$\langle 2g_i g_j, u g_1 g_3 + 2u g_1 \rangle, i \neq j = 1, 2, 3.$	3
$\langle 2g_i g_j, u g_1 g_3 + 2u g_3 \rangle, i \neq j = 1, 2, 3.$	3
$\langle 2g_i g_j, u g_1 g_3 + 2u \rangle, i \neq j = 1, 2, 3.$	3
$\langle 2g_i g_j, u g_2 g_3 + 2u g_2 \rangle, i \neq j = 1, 2, 3.$	1
$\langle 2g_i g_j, u g_2 g_3 + 2u g_3 \rangle, i \neq j = 1, 2, 3.$	1
$\langle 2g_i g_j, u g_2 g_3 + 2u \rangle, i \neq j = 1, 2, 3.$	1
$\langle 2g_i g_j, u g_1 + 2u \rangle, i \neq j = 1, 2, 3.$	6
$\langle 2g_i g_j, u g_2 + 2u \rangle, i \neq j = 1, 2, 3.$	4
$\langle 2g_i g_j, u g_3 + 2u \rangle, i \neq j = 1, 2, 3.$	4
$\langle 2g_i g_j, 3u \rangle, i \neq j = 1, 2, 3.$	7
$\langle 2g_i, u g_1 g_2 + 2u g_1 \rangle, i = 1, 2, 3.$	3
$\langle 2g_i, u g_1 g_2 + 2u g_2 \rangle, i = 1, 2, 3.$	3
$\langle 2g_i, u g_1 g_2 + 2u \rangle, i = 1, 2, 3.$	3
$\langle 2g_i, u g_1 g_3 + 2u g_1 \rangle, i = 1, 2, 3.$	3
$\langle 2g_i, u g_1 g_3 + 2u g_3 \rangle, i = 1, 2, 3.$	3
$\langle 2g_i, u g_1 g_3 + 2u \rangle, i = 1, 2, 3.$	3
$\langle 2g_i, u g_2 g_3 + 2u g_2 \rangle, i = 1, 2, 3.$	1
$\langle 2g_i, u g_2 g_3 + 2u g_3 \rangle, i = 1, 2, 3.$	1
$\langle 2g_i, u g_2 g_3 + 2u \rangle, i = 1, 2, 3.$	1
$\langle 2g_i, u g_1 + 2u \rangle, i = 1, 2, 3.$	6
$\langle 2g_i, u g_2 + 2u \rangle, i = 1, 2, 3.$	4
$\langle 2g_i, u g_3 + 2u \rangle, i = 1, 2, 3.$	4
$\langle 2g_i, 3u \rangle, i = 1, 2, 3.$	7
$\langle 2, u g_1 g_2 + 2u g_1 \rangle.$	3
$\langle 2, u g_1 g_2 + 2u g_2 \rangle.$	3
$\langle 2, u g_1 g_2 + 2u \rangle.$	3
$\langle 2, u g_1 g_3 + 2u g_1 \rangle.$	3
$\langle 2, u g_1 g_3 + 2u g_3 \rangle.$	3
$\langle 2, u g_1 g_3 + 2u \rangle.$	3
$\langle 2, u g_2 g_3 + 2u g_2 \rangle.$	1
$\langle 2, u g_2 g_3 + 2u g_3 \rangle.$	1
$\langle 2, u g_2 g_3 + 2u \rangle.$	1
$\langle 2, u g_1 + 2u \rangle.$	6
$\langle 2, u g_2 + 2u \rangle.$	4
$\langle 2, u g_3 + 2u \rangle.$	4
$\langle 2, 3u \rangle.$	7

Table 2. Non-zero cyclic codes of length 7 over $\mathbb{Z}_4 + u\mathbb{Z}_4, u^2 = 0$.

Non-zero generator polynomials	ranks
$\langle g_1g_2 + 2g_1, ug_1 + 2u \rangle .$	6
$\langle g_1g_2 + 2g_1, ug_2 + 2u \rangle .$	4
$\langle g_1g_2 + 2g_1, 3u \rangle .$	7
$\langle g_1g_2 + 2g_2, ug_1 + 2u \rangle .$	6
$\langle g_1g_2 + 2g_2, ug_2 + 2u \rangle .$	4
$\langle g_1g_2 + 2g_2, 3u \rangle .$	7
$\langle g_1g_2 + 2, ug_1 + 2u \rangle .$	6
$\langle g_1g_2 + 2, ug_2 + 2u \rangle .$	4
$\langle g_1g_2 + 2, 3u \rangle .$	7
$\langle g_1g_3 + 2g_1, ug_1 + 2u \rangle .$	6
$\langle g_1g_3 + 2g_1, ug_3 + 2u \rangle .$	4
$\langle g_1g_3 + 2g_1, 3u \rangle .$	7
$\langle g_1g_3 + 2g_3, ug_1 + 2u \rangle .$	6
$\langle g_1g_3 + 2g_3, ug_3 + 2u \rangle .$	4
$\langle g_1g_3 + 2g_3, 3u \rangle .$	7
$\langle g_1g_3 + 2, ug_1 + 2u \rangle .$	6
$\langle g_1g_3 + 2, ug_3 + 2u \rangle .$	4
$\langle g_1g_3 + 2, 3u \rangle .$	7
$\langle g_2g_3 + 2g_2, ug_2 + 2u \rangle .$	4
$\langle g_2g_3 + 2g_2, ug_3 + 2u \rangle .$	4
$\langle g_2g_3 + 2g_2, 3u \rangle .$	7
$\langle g_2g_3 + 2g_3, ug_2 + 2u \rangle .$	4
$\langle g_2g_3 + 2g_3, ug_3 + 2u \rangle .$	4
$\langle g_2g_3 + 2g_3, 3u \rangle .$	7
$\langle g_2g_3 + 2, ug_2 + 2u \rangle .$	4
$\langle g_2g_3 + 2, ug_3 + 2u \rangle .$	4
$\langle g_2g_3 + 2, 3u \rangle .$	7
$\langle g_1 + 2, 3u \rangle .$	7
$\langle g_2 + 2, 3u \rangle .$	7
$\langle g_3 + 2, 3u \rangle .$	7

References

- [1] T. Abualrub and R. Oehmke, On the Generators of \mathbb{Z}_4 Cyclic Codes of Length 2^e , *IEEE Trans. Inform. Theory*, **49**, 2003, 2126-2133.
- [2] B. Yildiz and S. Karadeniz, Linear codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$, MacWilliams identities, projections, and formally self-dual codes, *Finite Fields Appl.*, **27**, (2014), 24-40.

- [3] A. R. Hammons , Jr., P. V. Kumar, A. R. Calderbank, N. J. A. Sloane, P. Sole, The \mathbb{Z}_4 linearity of Kerdock, Preparata, Goethals and related codes, *IEEE Trans. Inform. Theory* **40(2)**, (1994), 301-319.
- [4] T. Abualrub, I. Siap, Cyclic codes over the rings $\mathbb{Z}_2 + u\mathbb{Z}_2$ and $\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$, *Designs, Codes and Cryptography*, **42**, (2007), 273-287.
- [5] S. X. Zhu and X. Kai, Dual and self-dual negacyclic codes of even length over \mathbb{Z}_{2^a} , *Discrete Mathematics*, **13**, (2008), 7-10.
- [6] A. K. Singh and P. K. Kewat, On cyclic codes over the ring $\mathbb{Z}_p[u]/\langle u^k \rangle$, *Des. Codes Cryptogr.*, DOI 10.1007/s10623-013-9843-2.
- [7] S. Ling, P. Sole , Type II codes over $F_4 + uF_4$, *European J. Combin*, **12**, (2001), 983-997.
- [8] S. T. Dougherty and K. Shiromoto, Maximum Distance Codes Over Rings of Order 4, *IEEE Trans. Inform. Theory*, **47**, (2001), 400-405.
- [9] Hai Quang Dinh and Sergio R. Lopez-Permouth, Cyclic and Negacyclic Codes Over Finite Chain Rings, *IEEE Transactions on Information Theory*, **50**, (2004), 1728-1744.
- [10] R. K. Bandi and M. Bhaintwal, Cyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$, arXiv: 1501.01327v1 [cs. IT] (2015).
- [11] S. T. Dougherty and S. Ling, Cyclic codes over \mathbb{Z}_4 of even length, *Des. Codes Cryptogr.*, **39**, (2006), 127-153.
- [12] J. Wolfmann, Negacyclic and cyclic codes over \mathbb{Z}_4 , *IEEE Trans. Inform. Theory*, **45(7)**, (1999), 2527-2532.
- [13] V. S. Pless and Z. Qian, Cyclic codes and quadratic residue codes over \mathbb{Z}_4 , *IEEE Trans. Inform. Theory*, **42(5)**, (1996), 1594-1600.
- [14] N. Aydin, D. Ray-Chaudhuri, Quasi-cyclic codes over \mathbb{Z}_4 and some new binary codes, *IEEE Transactions on Information Theory*, **48**, (2002), 2065-2069.
- [15] B. Yildiz, N. Aydin, On cyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ and their \mathbb{Z}_4 images, *Int. J. Information and Coding Theory*, **2(4)**, (2014), 226-237.